

# The Impact of Risk and Ambiguity Aversion on Annuity and Saving Choices

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## Abstract

We analyze the impact of risk and ambiguity aversion using a lifecycle recursive utility model. Both risk and ambiguity aversion are shown to reduce annuity demand and enhance bond holdings. We obtain this result using an intertemporal framework in which we can vary both risk and ambiguity aversion, while preserving preference monotonicity.

**Keywords:** recursive utility, lifecycle model, ambiguity aversion, risk aversion, saving choices, annuity puzzle.

**JEL codes:** D91, G11, J17.

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# 1 Introduction

Risk and ambiguity aversion are two central behavioral traits affecting economic lifecycle problems, such as saving and portfolio choices. A natural question is whether ambiguity and risk aversion have different impacts, or whether they generate qualitatively similar effects, especially when risk and ambiguity aversion are considered simultaneously. Although these traits appear to have certain similarities, with both traits aiming to model attitudes towards (objective or subjective) uncertainty, they have generally been studied in two separate strands of the economic literature. As pointed out in Guetlein (2016), the reason for the lack of joint analysis is that the problem is complicated and may in general lead to non-clear-cut results. Some insights have been given in static setups for portfolio choice problems (see e.g., Dow and Werlang 1992, Gollier 2011), and for self-insurance and self-protection questions (see among others, Treich 2010, Snow 2011, and Alary et al. 2013). However, the question has never really been theoretically addressed in intertemporal problems.

In this paper, we succeed in deriving non-ambiguous results regarding the joint role of risk and ambiguity aversion in a lifecycle model. A lifecycle model with uncertain lifetime is a natural workhorse for such an analysis. Not only is mortality a large risk in life but life expectancies are highly heterogeneous, as measured by so-called life disparity. Furthermore, even if life disparity can be partly explained by socio-economic factors, or differences in health systems, significant unexplained factors remain (Shkolnikov et al. 2003, Edwards and Tuljapurkar 2005, or Shkolnikov et al. 2011). Mortality can therefore be seen as both risky and ambiguous.

Formally, we focus on a smooth ambiguity model à la Hayashi and Miao (2011), which nests some standard models such as that of Klibanoff et al. (2009) or the recursive specification of Epstein and Zin (1989). We analyze the saving decisions of an agent who is both ambiguity and risk averse, and who may invest both in risk-free bonds and in annuities. Mortality is the sole risk faced by the agent and she can live at most for two periods. The key feature of our approach is that we additionally assume that the agent's preferences are monotone with respect to first-order stochastic dominance. This property enables us to jointly characterize the respective roles of ambiguity and risk aversion.

What does the monotonicity property imply? In a nutshell, monotonicity prevents agents from opting for dominated choices. More precisely, if two choices are available, with the first one yielding preferred outcomes in all circumstances, then an agent with monotone preferences will always prefer the first choice to the second. This property has already been studied in a risk setting. Bommier et al. (2017b) show that the only Kreps and Porteus (1978) monotone preferences able to disentangle risk aversion and intertemporal elasticity of substitution are the so-called risk-sensitive preferences introduced by Hansen and Sargent (1995). These preferences have proved to be useful for characterizing the role of risk aversion for various consumption-saving problems. For instance, in a very general infinite-horizon setting, Bommier and Le Grand (2019) show that once monotonicity is imposed, risk aversion unambiguously increases precautionary savings.

While the impact of monotonicity has already been explored in a risk setting, we are not aware of any extension to an uncertainty setting, apart from the representation results provided in Bommier and Le Grand (2014a) and Bommier et al. (2017b). In particular, no application has been developed, leaving the practical implications of using monotone preferences in an ambiguity setting unclear. The current paper fills this gap, showing that monotone preferences can be used to derive clear-cut and intuitive predictions regarding the impact of risk and ambiguity aversion on savings and annuity purchases. More precisely, we prove that a higher ambiguity aversion, while maintaining risk aversion constant, leads to higher holdings of riskless bonds but smaller holdings of annuities.<sup>1</sup> There are different ways to define what is an increase in risk aversion. An increase in risk aversion in the sense introduced by Guetlein (2016) – which does not preserve ambiguity attitudes – typically leads to non-clear-cut results. However, a compensated or “net” change in risk aversion, where the ambiguity parameter is modified together with the risk aversion parameter so as to keep ambiguity attitudes unchanged, is shown to have unambiguous implications, with an impact on asset demands similar to that of an increase in ambiguity aversion. Our results are consistent with intuition, since in the presence of mortality uncertainty alone, the annuity – which only pays

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<sup>1</sup>This result holds for interior solutions, where agents purchase positive quantities of bonds and annuities. We also derive slightly different results for corner solutions, where either annuity or bond holdings hit non-negativity constraints and are equal to zero.

off in *good* states of the world, when the agent remains alive – is a risk-taking device. Unsurprisingly, agents exhibiting greater ambiguity aversion or greater (net) risk aversion are less willing to take a bet on their survival through annuity saving, and therefore invest fewer resources in annuities.

Interestingly, our results tend to indicate that ambiguity and risk aversion have, qualitatively speaking, very similar impacts. In other words, what matters is the degree of uncertainty aversion (encompassing both risk and ambiguity aversion), rather than whether uncertainty is objective or subjective. Of course, quantitative results are likely to depend on whether uncertainty is objective or subjective, and on whether agents exhibit greater or smaller aversion for objective and subjective uncertainty. Nonetheless, the overall message is that greater uncertainty aversion (be it greater risk aversion or greater ambiguity aversion) tends to reduce annuity purchases and enhance investments in bonds. Uncertainty aversion in particular appears to be a natural candidate for explaining the low annuitization level observed in the data (see Johnson et al. 2004 for empirical evidence in the US). This confirms the early findings of Bommier and Le Grand (2014b), who examined risk aversion, or those of d’Albis and Thibault (2018), who focused on ambiguity aversion in a static, one-period model.

To the best of our knowledge, this is the first paper to derive clear-cut results regarding the joint impact of risk and ambiguity aversion in an intertemporal framework. Our article obviously connects to the literature that discusses the roles of risk and ambiguity aversion separately. In the risk setting, there is an abundant literature on precautionary savings (see, among others, Drèze and Modigliani 1972, Kimball 1990, Bleichrodt and Eeckhoudt 2005, Courbage and Rey 2007, Kimball and Weil 2009, Jouini et al. 2013, Nocetti 2016) and on annuity choices (Yaari 1965, Davidoff et al. 2005, Lockwood 2012, Pashchenko 2013, among many others). The role of risk aversion is, however, not often studied because most of the literature relies on the standard time-additive setup, which makes it impossible to isolate the role of risk aversion, since intertemporal elasticity of substitution and risk aversion are intertwined. Among the exceptions, we find papers that use recursive frameworks, such as van der Ploeg (1993), Weil (1993), Kimball and Weil (2009), and several others. Nevertheless, most of these contributions rely on non-monotone

preferences (notably those using the most popular Epstein-Zin specification with an intertemporal elasticity of substitution different from 1), with recursivity and monotonicity being combined in very few papers, including van der Ploeg (1993), Tallarini (2000) and Bommier and Le Grand (2014b, 2019). None of these articles feature ambiguity aversion.

In an ambiguity setting, intertemporal problems are typically addressed using recursive extensions of standard static ambiguity models, though without having enough flexibility to change both risk and ambiguity aversion. We note, among others, the analyses of Osaki and Schlesinger (2014), Berger (2014), and Kajii and Xue (2016), who examine the precautionary savings of ambiguity averse agents, or the study of Collard et al. (2018), who investigate whether ambiguity aversion can explain historical values of the equity premium. All of these papers focus on ambiguity aversion, while restricting their analyses to models that reduce to the standard time-additive model when uncertainty is purely objective (in cases where there is no ambiguity). In other words, they retain the lack of flexibility of the standard time-additive model, which makes it impossible to explore the role of risk aversion. An exception is Peter (2019), who considers a more flexible framework, relying on a non-separable, two-period utility function, although the author does not investigate the role of risk aversion. The current paper introduces flexibility by using the recursive framework of Hayashi and Miao (2011), a route also followed by more quantitatively oriented papers (see e.g., Backus et al. 2015). Our contribution is notable for its use of monotone preferences, which in addition to being an intuitive assumption, affords great tractability and the ability to derive formal results.

## **2 Risk and ambiguity aversions in intertemporal frameworks**

### **2.1 The recursive smooth ambiguity model**

To study the effect of risk and ambiguity aversion on the optimal decisions of an agent who faces an ambiguous mortality risk, we adopt the recursive smooth

ambiguity utility model axiomatized by Hayashi and Miao (2011), in which Seo (2009)'s construction of a static smooth ambiguity model using Anscombe and Aumann (1963) acts is embedded into an infinite horizon setting.

Let  $\Omega$  be the finite state space and let  $\mathcal{X}$  be the set of consequences. The set of Anscombe-Aumann acts is  $\mathcal{G}_0 \equiv (\Delta(\mathcal{X}))^\Omega$ , where  $\Delta(\mathcal{X})$  denotes the set of Borel probability measures defined on the Borel subsets of  $\mathcal{X}$ . We refer to Hayashi (2005) for the formal construction of the domain of *compound lottery-acts*, which are defined as the subset of dynamically coherent acts in the product space  $\prod_{t=0}^{\infty} \mathcal{G}_t$ , where the spaces  $\mathcal{G}_t$  are defined by the recursion  $\mathcal{G}_t = (\Delta(\mathcal{X} \times \mathcal{G}_{t-1}))^\Omega$ .

The representation result of Hayashi and Miao (2011) states that the following exist: an aggregator  $W: \mathcal{X} \times \mathbb{R} \rightarrow \mathbb{R}$ , continuous and increasing in its second argument; two continuous and strictly increasing functions  $\psi_A, \psi_R: \mathbb{R} \rightarrow \mathbb{R}$ ; and a probability measure  $\mu \in \Delta(\Delta(\Omega))$  over the set of objective distributions, such that the utility of  $(x, g) \in \mathcal{X} \times \mathcal{G}$  is given by:

$$V(x, g) = W \left( x, \psi_A^{-1} \left( \int_{\Delta(\Omega)} \psi_A \circ \psi_R^{-1} \left( \sum_{\omega \in \Omega} \pi(\omega) \right. \right. \right. \quad (1) \\ \left. \left. \left. \times \int_{\mathcal{C} \times \mathcal{G}} \psi_R(V(x', g')) g(\omega) (dc', dg') \right) \mu(d\pi) \right) \right).$$

Preferences exhibit ambiguity aversion if  $\psi_A \circ \psi_R^{-1}$  is concave.

## 2.2 Monotone preferences

For our analysis, we will additionally require preferences to be monotone with respect to first order stochastic dominance. Loosely speaking, monotonicity property means that if an agent prefers the outcomes of a given action to those of another action in all states of the world, she should always prefer the former action to the latter. This property is a consistency requirement between preferences for deterministic outcomes, subjective beliefs, and preferences in the presence of risk and ambiguity.

Monotonicity has already been analyzed in risk settings. It has straightforward implications for saving behaviors. Consider, for instance, an agent who is likely to live for one or two periods. With monotone preferences, her savings in the

presence of mortality risk will be bound by her savings when she is sure to live for either one or two periods. These savings will, in particular, be smaller than her savings when she is sure to live for two periods. With non-monotone preferences, and for certain parameterizations, the agent is likely to save *more* in the presence of mortality risk than she would save knowing that she is sure to live for two periods.<sup>2</sup> Aside from this implication regarding saving behaviors, monotonicity imposes strong restrictions on preference representations, as shown in Bommier et al. (2017b). This latter contribution shows, in particular, that risk-sensitive preferences are the only Kreps and Porteus (1978) monotone preferences able to disentangle risk aversion and the elasticity of intertemporal substitution. Because of monotonicity, these preferences may yield clear-cut insights into the role of risk aversion (see Bommier and Le Grand 2019 for precautionary savings).

In the setting of Hayashi and Miao (2011), monotonicity requires the representation in (1) to have an affine aggregator  $W$  and functions  $\psi_R$  and  $\psi_A$  to be of the “constant absolute risk aversion” kind. See Bommier and Le Grand (2014a) for a formal proof. The specification (1) can then be written as:

$$V(x, g) = f(x) - \frac{\beta}{k_A} \log \left( \int_{\Delta(\Omega)} \exp \left( \frac{k_A}{k_R} \log \left( \sum_{\omega \in \Omega} \pi(\omega) e^{-k_R V} \right) \right) \mu(d\pi) \right), \quad (2)$$

where  $f : \mathcal{X} \rightarrow \mathbb{R}_+$  is the instantaneous utility function. The parameters  $k_A$  and  $k_R$  drive ambiguity and risk aversion. Formally, two agents are comparable in terms of ambiguity aversion if they only differ by the parameter  $k_A$ , the agent with greater  $k_A$  being more ambiguity averse. Two agents are comparable in terms of risk aversion if they only differ by the parameter  $k_R$ , the agent with greater  $k_R$  being more risk averse. It is therefore natural to refer to  $k_A$  as being the ambiguity aversion parameter, and to  $k_R$  as being the risk aversion parameter.

However, as emphasized by Guetlein (2016), the smooth model does not offer a straightforward separation between ambiguity and risk attitudes. In particular, whether the agent is ambiguity averse or ambiguity lover depends on whether the ratio  $\frac{k_A}{k_R}$  is greater or smaller than one. Ambiguity neutrality is obtained when this ratio is equal to one. It follows that increasing the risk aversion parameter  $k_R$  while keeping  $k_A$  constant will change ambiguity attitudes. For example, an agent

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<sup>2</sup>See Bommier et al. (2017a) for a detailed example.

who is more risk averse than an agent who is ambiguity neutral will necessarily be ambiguity lover (see discussion in Section II of Guetlein).

To avoid such shortcomings, Guetlein (2016) suggests considering a simultaneous increase in both risk aversion and ambiguity aversion that leaves ambiguity attitudes unchanged (see Section III of her paper). In the general framework of Guetlein (2016), such joint increases can be rather complex, involving different transformations of the risk aversion and ambiguity aversion functions. Under the assumption of preference monotonicity considered in the current paper, the situation is, however, much simpler. Increasing uncertainty aversion – to use Guetlein’s terminology – while preserving ambiguity attitudes, is obtained by increasing both  $k_A$  and  $k_R$ , while keeping the ratio  $\frac{k_A}{k_R}$  constant. To avoid confusion, we will introduce the terminology “net” vs. “gross” to distinguish two possible forms of the increase in risk aversion. By definition, an increase in “gross risk aversion” involves increasing  $k_R$  while keeping  $k_A$  constant. An increase in “net risk aversion” involves increasing  $k_A$  and  $k_R$ , while keeping the ratio  $\frac{k_A}{k_R}$  unchanged, so as to preserve ambiguity attitudes. Here, the adjective “net” refers to an approach where risk aversion would be defined residually by looking at the difference (in fact the ratio here) between uncertainty aversion and ambiguity aversion.

### 3 Results in a two-period framework

#### 3.1 Preference specifications

We use the above setting to study the effect of risk and ambiguity aversion on the optimal annuitization of an agent who faces an ambiguous mortality risk.

There are two periods, indexed by  $t = 1, 2$ . The agent is alive in the first period, and can be either dead or alive in the second period. When alive, the agent consumes, when dead she may bequeath wealth to her heirs. The formal connection between our two-period setting and the infinite horizon setting of Hayashi and Miao (2011) can be made by assuming that the first two periods are always followed by an infinite sequence of periods where the agent is dead and bequeaths nothing. In the second period, the state of the agent can be described by two variables: (i) a dummy variable  $d \in \{alive, dead\}$ , indicating whether the agent is alive



or dead, and (ii) a real number indicating how much she consumes (if alive) or bequeaths (if dead). Thus, formally, the set of consequences for the second period is  $\mathcal{X} = \{alive, dead\} \times \mathbb{R}_+$ . Because of our assumption that the agent is always alive in period 1, the first-period set of consequences is simply  $\{alive\} \times \mathbb{R}_+$  and the domain of choice is  $(\{alive\} \times \mathbb{R}_+) \times (\Delta(\mathcal{X}))^\Omega$ , where  $\Omega$  is a finite subjective state-space.

This can be simplified since we assume that mortality is the only source of uncertainty. The agent's subjective prior can then be described by a probability distribution over survival probabilities, i.e. a list of pairs  $(m_i, p_i) \in [0, 1]^2$  for  $i = 1, \dots, n$  such that  $\sum_{i=1}^n m_i = 1$ . The interpretation is that there are  $n$  subjective states and that the agent (subjectively) associates the probability  $m_i$  with the state  $i \in \{1, \dots, n\}$  that corresponds to a survival probability equal to  $p_i$  (and a mortality probability equal to  $1 - p_i$ ). A setting with no ambiguity is one in which  $m_i = 1$  for some  $i$  and  $m_j = 0$  for all  $j \neq i$ : The agent is certain to survive with probability  $p_i$ . Conversely, ambiguity occurs whenever there are two states  $i$  and  $j$  with  $p_i \neq p_j$  and  $m_i, m_j \neq 0$ .

The instantaneous utility function  $f$  that appears in specification (2) is formally defined on  $\mathcal{X} = \{alive, dead\} \times \mathbb{R}_+$ . In other words,  $f : (d, x) \in \{alive, dead\} \times \mathbb{R}_+ \mapsto f(d, x) \in \mathbb{R}$ . To simplify the notation, rather than introducing the dummy variable  $d$  to indicate whether the agent is alive or dead, we will simply use different letters  $c$  and  $w$  to indicate whether it refers to a level of consumption or a bequest level. A number  $c \in \mathbb{R}_+$  must therefore be interpreted as an element  $(alive, c) \in \mathcal{X}$ , while a number  $w \in \mathbb{R}_+$  must be interpreted as an element  $(dead, w) \in \mathcal{X}$ . We will introduce the notation  $u(c) = f(alive, c)$  for the instantaneous utility of consumption (if alive) and  $v(w) = f(dead, w)$  for the instantaneous utility of bequest (if dead). As is standard, functions  $u$  and  $v$  are both assumed to be twice continuously differentiable, strictly increasing, and concave. In representation (2), the instantaneous utility  $f$  is defined up to a constant and we can assume without loss of generality that  $v(0) = 0$ . Note that once this normalization has been adopted, adding or subtracting a constant to  $u$  cannot then be seen as mere normalization. The utility gap  $u(c) - v(w)$  in fact measures how much the agent values being alive and consuming  $c$  versus being dead and bequeathing wealth  $w$ . The larger

the utility gap, the more attractive being alive and consuming becomes compared to being dead and bequeathing. The baseline situation corresponds to the one where the agent exhibits no altruism towards her heirs, that is to the case in which  $v$  is constant (and thus equal to 0).

More generally, the utility gap  $u - v$  closely relates to the *value of life* concept, which quantifies how much an agent is willing to pay for increasing her survival probabilities. As shown in Bommier et al. (2017b), the sign of the value of life is crucial for understanding the role of risk aversion on savings. There is a wide empirical literature on value of life estimates, both from academics (see for instance Viscusi and Aldy 2003 for a survey of value-of-life estimates throughout the world) and from institutions (see the US Environmental Protection Agency Office of Air and Radiation 2011 report, for instance). Despite the heterogeneity in estimates, there is a strong consensus that the value of life is positive and large. For instance, a reasonable estimate for the US is around 7 million US dollars.

Using the above notation and representation (2), the utility of an agent who consumes  $c_1$  in period 1 and consumes  $c_2$  or bequeaths  $w$  in period 2 (depending on whether she survives or not) is given by:

$$U(c_1, c_2, w) = u(c_1) - \frac{\beta}{k_A} \log \left( \sum_{i=1}^n m_i \left( p_i e^{-k_R u(c_2)} + (1 - p_i) e^{-k_R v(w)} \right)^{\frac{k_A}{k_R}} \right). \quad (3)$$

The utility specification  $U(c_1, c_2, w)$  in equation (3) is the objective function that the agent maximizes, subject to the budget constraints that will be introduced later on. We will derive comparative statics results related to the ambiguity and risk aversion parameters  $k_A$  and  $k_R$ , while assuming that individuals' beliefs (the  $m_i$  and  $p_i$ ) are unchanged.

### 3.2 Particular cases

The general preference representation (3) embeds several particular cases that we describe below.

**No ambiguity.** The no-ambiguity case corresponds to  $m_{i_0} = 1$  for some  $i_0$  and  $m_i = 0$  for  $i \neq i_0$ . The preference representation (3) becomes:

$$U(c_1, c_2, w) = u(c_1) - \frac{\beta}{k_R} \log \left( p_{i_0} e^{-k_R u(c_2)} + (1 - p_{i_0}) e^{-k_R v(w)} \right), \quad (4)$$

which corresponds to the risk-sensitive preferences of Hansen and Sargent (1995). In particular, the parameter  $k_A$  no longer plays a role and risk aversion is driven by the parameter  $k_R$ , where more risk averse agents correspond to larger values of the parameter  $k_R$ .

The case  $k_R = 0$ , can be obtained by taking the limit of expression (4) when  $k_R \rightarrow 0$ , and corresponds to the standard additive model where  $U(c_1, c_2, w) = u(c_1) + \beta (p_{i_0} u(c_2) + (1 - p_{i_0}) v(w))$ .

**Ambiguity neutrality.** Ambiguity neutrality corresponds to  $k_A = k_R$ . The utility expression (3) becomes:

$$U(c_1, c_2, w) = u(c_1) - \frac{\beta}{k_R} \log \left( p e^{-k_R u(c_2)} + (1 - p) e^{-k_R v(w)} \right), \quad (5)$$

where  $p = \sum_{i=1}^n m_i p_i$  is the average mortality probability. Formally, this setup is similar to the no-ambiguity case and expression (5) looks very much like expression (4). Again, when  $k_R = 0$ , we fall back on the additive model.

**Temporal risk neutrality.** The case  $k_R = 0$  implies temporal risk neutrality. By continuity, for  $k_R \rightarrow 0$  from expression (3), we deduce that the utility expression becomes:

$$U(c_1, c_2, w) = u(c_1) - \frac{\beta}{k_A} \log \left( \sum_{i=1}^n m_i \exp \left( -k_A (p_i u(c_2) + (1 - p_i) v(w)) \right) \right).$$

This fits into the Klibanoff et al. (2009) recursive framework and exactly corresponds to the specification in Collard et al. (2018).

**No objective probabilities.** The last case corresponds to a setup where all information is subjective. In other words, a situation where, for all  $i$ , we have either  $p_i = 0$  or  $p_i = 1$ . This situation reduces to a two (subjective)-state model, where  $p_1 = 0$  occurs with subjective probability  $m_1$  and  $p_2 = 1$  with subjective

probability  $m_2 = 1 - m_1$ . We then have:

$$U(c_1, c_2, w) = u(c_1) - \frac{\beta}{k_A} \log \left( m_1 e^{-k_A u(c_2)} + (1 - m_1) e^{-k_A v(w)} \right),$$

similar to expressions (4) and (5), with  $k_A$  instead of  $k_R$ .

### 3.3 The agent program

We consider an ambiguity-and-risk-averse agent, whose preferences are assumed to be represented by the utility function defined in equation (3). In the first period, the agent is endowed with an initial wealth  $W > 0$ . She has no other source of income during the two periods but can transfer consumption from the first period to the second through savings in a bond and in an annuity. These savings are denoted by  $s$  and  $a$ , respectively, and the first period budget constraint is simply  $W = c_1 + s + a$ . The bond pays off the riskless gross interest rate  $r > 0$  in the second period, regardless of whether the agent is alive or not. Bond payoffs, in particular, can be bequeathed to the agent's heirs. The annuity pays off the gross rate  $r_a > r$ , which is assumed to be higher than the riskless rate. Annuity payoffs are only made in cases of survival. This means that the second period budget constraints are  $c_2 = r_a a + rs$  when the agent lives for two periods and consumes the outcomes of her savings and  $w = rs$  when the agent dies and bequeaths her bond savings.

We make two observations regarding the annuity return  $r_a$ . First, in our ambiguity setting, the notion of an actuarially fair annuity is not a very relevant concept. An objective survival probability can still be defined, as can an objective definition of actuarially fair annuities. However, from the agent's perspective, what actually matters is a subjective notion of fairness, which is not generally unequivocally defined due to the ambiguity setting. We therefore define the annuity return simply by using the gross rate  $r_a$ , higher than  $r$ , instead of defining it using the riskless interest rate and some survival probability. The second observation is that the annuity return rate  $r_a$  is not affected by ambiguity and is assumed to be perceived as constant by the agent. This implicitly means that the return  $r_a$  actually comes from pooling the heterogeneous mortality risk, such that the agent cannot infer her objective survival probability from the gross rates  $r_a$  and

$r$ . Another (nonexclusive) possibility would be to assume that the annuity return embeds a fee that cannot be observed by the agent.

Using the previous notation, the agent's program can therefore be written as:

$$\max_{(s,a) \in \mathbb{R}^2} u(W - a - s) - \frac{\beta}{k_A} \log \left( \sum_{i=1}^n m_i \left( p_i e^{-k_R u(r_a a + r s)} + (1 - p_i) e^{-k_R v(r s)} \right)^{\frac{k_A}{k_R}} \right), \quad (6)$$

$$\text{s.t. } W - a - s \geq 0, \quad s \geq 0, \quad a \geq 0, \quad (7)$$

where we exclude negative consumption levels, borrowings, and annuity short-selling.

### 3.4 The role of risk and ambiguity aversion

The impact of  $k_A$  and  $k_R$  on saving choices is summarized in the following proposition, whose proof is given in the Appendix.

**Proposition 1** *Consider the consumption-saving program of equations (6) and (7) with  $k_A \geq k_R > 0$ . We denote by  $a(k_A, k_R) \geq 0$  and  $s(k_A, k_R) \geq 0$  the optimal savings in annuities and bonds, respectively.*

*We assume that the optimum is characterized by a positive value of life, i.e., formally,  $u(r_a a(k_A, k_R) + r s(k_A, k_R)) > v(r s(k_A, k_R))$ .*

*We distinguish three cases (besides the trivial case  $a(k_A, k_R) = s(k_A, k_R) = 0$ ).*

1. *Case  $a(k_A, k_R) > 0$  and  $s(k_A, k_R) > 0$ .*

(a) *Ambiguity aversion. We have:  $\frac{\partial a(k_A, k_R)}{\partial k_A} \Big|_{k_R} < 0$ ,  $\frac{\partial s(k_A, k_R)}{\partial k_A} \Big|_{k_R} > 0$ , as well as  $\frac{\partial (r_a a(k_A, k_R) + r b(k_A, k_R))}{\partial k_A} \Big|_{k_R} < 0$ .*

(b) *Net risk aversion. We have:  $\frac{\partial a(k_A, k_R)}{\partial k_R} \Big|_{k_A/k_R} < 0$ ,  $\frac{\partial s(k_A, k_R)}{\partial k_R} \Big|_{k_A/k_R} > 0$ ,  $\frac{\partial (r_a a(k_A, k_R) + r b(k_A, k_R))}{\partial k_R} \Big|_{k_A/k_R} < 0$ .*

2. *Case  $a(k_A, k_R) > 0$  and  $s(k_A, k_R) = 0$ :  $\frac{\partial a(k_A, k_R)}{\partial k_A} \Big|_{k_R} < 0$  and  $\frac{\partial a(k_A, k_R)}{\partial k_R} \Big|_{k_A/k_R} < 0$ .*

3. *Case  $a(k_A, k_R) = 0$  and  $s(k_A, k_R) > 0$ :  $\frac{\partial s(k_A, k_R)}{\partial k_A} \Big|_{k_R}$  and  $\frac{\partial s(k_A, k_R)}{\partial k_R} \Big|_{k_A/k_R}$  have the same sign as  $(v'(r s(k_A, k_R)) - u'(r s(k_A, k_R)))$ .*

The proposition characterizes the role of ambiguity and net risk aversion for an ambiguity-and-risk averse agent ( $k_A \geq k_R > 0$ ). It distinguishes three cases

depending on whether saving is constrained to one instrument. The interior case, corresponding to unconstrained annuity and bond holdings, is reported in case 1. Varying the ambiguity aversion parameter  $k_A$  – while keeping the risk aversion parameter  $k_R$  fixed – has an unambiguous impact on security holdings. A larger ambiguity aversion diminishes the demand for annuities but increases the demand for bonds. The overall impact is such that second period consumption (equal to the quantity  $r_a a(k_A, k_R) + r b(k_A, k_R)$ ) also diminishes when ambiguity aversion rises. The only condition required to derive these results is that, at the optimum, the utility of being alive in the second period and consuming is greater than the utility of being dead and bequeathing one’s wealth. In other words, the condition simply means that life is worth living, or that the value of life is positive. As already mentioned, despite some measurement heterogeneity, there is plenty of empirical evidence that the value of life is large and positive. This assumption is therefore reasonable and not very restrictive.

The role of the “net” risk aversion parameter is obtained by increasing  $k_R$  and  $k_A$  while maintaining a constant ratio  $k_A/k_R$ . We find that the impact of an increase in net risk aversion is qualitatively similar to that of an increase in ambiguity aversion.

Case 2, which corresponds to constrained bond savings, is very similar to case 1. Since only one saving instrument is available and no portfolio choice is possible, the result is much simpler to state. A higher ambiguity aversion – or a higher net risk aversion – implies smaller annuity holdings.

Case 3 corresponds to constrained annuity holdings. The result, be it for the role of ambiguity aversion or the role of net risk aversion, depends on the sign of the difference between the marginal utility of survival and the marginal utility of bequest. The intuition is as follows. An increase in ambiguity aversion – or in net risk aversion – translates into the willingness to increase lifetime utility in the bad state (i.e., death). With monotone preferences, this necessarily reduces the lifetime utility in the good state (i.e., survival) and diminishes the dispersion in lifetime utilities. In the absence of an annuity holding, the dispersion in lifetime utilities amounts to  $\beta(u(rs) - v(rs))$ , which is positive because the value of life is positive. The way in which the riskless saving  $s$  should be varied to reduce the dispersion

in lifetime utilities therefore depends on whether  $u(rs) - v(rs)$  is increasing or decreasing with  $s$  and thus on the sign of the derivative  $u'(rs) - v'(rs)$ . When the marginal utility in the good state is higher than the marginal utility in the bad state ( $u'(rs) - v'(rs) > 0$ ), reducing the dispersion in lifetime utilities will involve smaller savings. Conversely, if  $u'(rs) - v'(rs) < 0$ , an increase in ambiguity aversion or in net risk aversion will increase savings.

In the absence of a bequest motive ( $v' = 0$ ), the sign of the derivatives in case 3 is clear, and an increase in either net risk aversion or ambiguity aversion diminishes both savings and second period consumption. With non-trivial bequest motives, a very common specification (up to a normalization constant) is  $v(w) = \theta(u(\underline{w} + w) - u(\underline{w}))$ , where  $\theta \geq 0$  reflects the intensity of the altruistic motive, and  $\underline{w} > 0$  makes a bequest a luxury good. Furthermore, the intertemporal elasticity of substitution is also often assumed to be constant, with  $u'(c) = c^{-\sigma}$ .<sup>3</sup> In such a case, the sign of  $u'(rs) - v'(rs)$  is not clear-cut: it is typically positive for small values of  $s$  and becomes negative for larger values. The impact of ambiguity and net risk aversion will then depend on this sign, as shown in the proposition.

## 4 Discussion

We have proved that the comparative statics of ambiguity and net risk aversion yield non-ambiguous results in an ambiguity setting à la Anscombe-Auman. A larger ambiguity aversion or a larger net risk aversion means a smaller demand for annuities and a larger demand for riskless bonds. To obtain the intuition, note that annuities are uncertain assets that pay off in the good state of the world (i.e., when the agent lives a long life) and pay nothing in the bad state (i.e., in case of an early death). By analogy with finance, we can therefore consider that an annuity has the fundamental properties of a pro-cyclical asset. It is then quite natural that uncertainty aversion, be it ambiguity or risk aversion, reduces investment in such an asset. We would of course have found a converse result if we had considered hedging assets (such as life-insurance products that only pay in the case of death).

While our results may look intuitive, we emphasize that they go against the

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<sup>3</sup>Such calibrations can be found, for instance, in De Nardi (2004), De Nardi et al. (2010), Ameriks et al. (2011), Lockwood (2012), or Pashchenko (2013), among many others.

common claim that annuities being insurance products, the demand for annuities should increase with uncertainty aversion. Since mortality risk makes preferences state-dependent, the link between insurance demand and risk aversion is not that obvious. As our results indicate, there are cases where the demand for insurance may decrease (and not increase) with risk aversion.

Preference monotonicity is the key ingredient as it enables us to highlight the fundamental assumptions required for our analysis, and in particular the assumptions related to the value of life. Because of the monotonicity assumption, we may view the agent’s decision as being a trade-off between welfares in different states (without monotonicity the agent may want to reduce welfare in all states). Uncertainty aversion then has a fairly clear role, dictating how much weight to put on bad states compared to good ones. To derive the impact of an increase in uncertainty aversion it is therefore necessary to properly identify good and bad states – or in our framework to identify whether survival should be viewed as a good or a bad realization. In the tradition of revealed preferences, this involves looking at the literature on endogenous mortality, which clearly indicates that the value of life is positive.

A final take-home message is that to make predictions regarding the impact of uncertainty aversion in the presence of an exogenous, non-monetary background risk (mortality risk in our paper, although this could also be health risk, for instance), it is necessary to use information that can only be gleaned from decisions where the degree of background risk is endogenous.

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# Appendix: Proof of Proposition 1

## A Proof: Interior case $a, s > 0$

The program can be written as:

$$\max_{(s,a) \in \mathbb{R}_+^2} u(W - a - s) - \frac{\beta}{k_A} \log \left( \sum_{i=1}^n m_i \left( p_i e^{-k_R u(r_a a + rs)} + (1 - p_i) e^{-k_R v(rs)} \right)^{\frac{k_A}{k_R}} \right).$$

Assuming an interior equilibrium, the first-order conditions are:

$$u'(W - a - s) = \beta r_a u'(r_a a + rs) \frac{\sum_{i=1}^n m_i p_i \left( p_i + (1 - p_i) e^{k_R(u(r_a a + rs) - v(rs))} \right)^{\frac{k_A}{k_R} - 1}}{\sum_{i=1}^n m_i \left( p_i + (1 - p_i) e^{k_R(u(r_a a + rs) - v(rs))} \right)^{\frac{k_A}{k_R}}}, \quad (8)$$

$$u'(W - a - s) = \beta r u'(r_a a + rs) \frac{\sum_{i=1}^n m_i p_i \left( p_i + (1 - p_i) e^{k_R(u(r_a a + rs) - v(rs))} \right)^{\frac{k_A}{k_R} - 1}}{\sum_{i=1}^n m_i \left( p_i + (1 - p_i) e^{k_R(u(r_a a + rs) - v(rs))} \right)^{\frac{k_A}{k_R}}} \quad (9)$$

$$+ \beta r v'(rs) e^{k_R(u(r_a a + rs) - v(rs))} \frac{\sum_{i=1}^n m_i (1 - p_i) \left( p_i + (1 - p_i) e^{k_R(u(r_a a + rs) - v(rs))} \right)^{\frac{k_A}{k_R} - 1}}{\sum_{i=1}^n m_i \left( p_i + (1 - p_i) e^{k_R(u(r_a a + rs) - v(rs))} \right)^{\frac{k_A}{k_R}}}.$$

We introduce the following notation:

$$\eta_i = p_i + (1 - p_i) e^{k_R(u(r_a a + rs) - v(rs))}, \quad (10)$$

as well as the expectation  $E_m[\cdot]$  and the covariance  $Cov_m$  (when well-defined):

$$E_m[X] = \frac{\sum_{i=1}^n m_i \eta_i^{\frac{k_A}{k_R}} X_i}{\sum_{i=1}^n m_i \eta_i^{\frac{k_A}{k_R}}}, \quad (11)$$

$$Cov_m(X, Y) = E_m[XY] - E_m[X]E_m[Y], \quad (12)$$

Using this notation, the first-order condition (8) and (9) become:

$$u'(W - a - s) = \beta r_a u'(r_a a + rs) E_m \left[ \frac{p}{\eta} \right], \quad (13)$$

$$\left( 1 - \frac{r}{r_a} \right) u'(W - a - s) = \beta r v'(rs) e^{k_R(u(r_a a + rs) - v(rs))} E_m \left[ \frac{1 - p}{\eta} \right]. \quad (14)$$

## A.1 Impact of $k_A$

### A.1.1 First-order condition (13)

Computing the derivative of (13) with respect to  $k_A$  yields, using the notation  $\eta_i$  defined in (10):

$$\begin{aligned}
& -u''(W - a - s) \left( \frac{\partial a}{\partial k_A} + \frac{\partial s}{\partial k_A} \right) = \beta r_a \left( r_a \frac{\partial a}{\partial k_A} + r \frac{\partial s}{\partial k_A} \right) u''(r_a a + rs) \frac{\sum_{i=1}^n m_i p_i \eta_i^{\frac{k_A}{k_R} - 1}}{\sum_{i=1}^n m_i \eta_i^{\frac{k_A}{k_R}}} \\
& + \beta r_a u'(r_a a + rs) \left( u'(r_a a + rs) \left( r_a \frac{\partial a}{\partial k_A} + r \frac{\partial s}{\partial k_A} \right) - v'(rs) r \frac{\partial s}{\partial k_A} \right) e^{k_R(u(r_a a + rs) - v(rs))} \\
& \times \left( (k_A - k_R) \frac{\sum_{i=1}^n m_i p_i (1 - p_i) \eta_i^{\frac{k_A}{k_R} - 2}}{\sum_{i=1}^n m_i \eta_i^{\frac{k_A}{k_R}}} - k_A \frac{\sum_{i=1}^n m_i (1 - p_i) \eta_i^{\frac{k_A}{k_R} - 1} \sum_{i=1}^n m_i p_i \eta_i^{\frac{k_A}{k_R} - 1}}{(\sum_{i=1}^n m_i \eta_i^{\frac{k_A}{k_R}})^2} \right) \\
& + \frac{\beta r_a}{k_R} u'(r_a a + rs) \left( \frac{\sum_{i=1}^n m_i p_i \log(\eta_i) \eta_i^{\frac{k_A}{k_R} - 1}}{\sum_{i=1}^n m_i \eta_i^{\frac{k_A}{k_R}}} - \frac{\sum_{i=1}^n m_i p_i \eta_i^{\frac{k_A}{k_R} - 1} \sum_{i=1}^n m_i \log(\eta_i) \eta_i^{\frac{k_A}{k_R}}}{\sum_{i=1}^n m_i \eta_i^{\frac{k_A}{k_R}} \sum_{i=1}^n m_i \eta_i^{\frac{k_A}{k_R}}} \right).
\end{aligned}$$

Introducing the operator  $E_m$  of equation (11), we obtain after dividing both sides of equality by  $\beta r_a u'(r_a a + rs)$  and using (13):

$$\begin{aligned}
& -E_m \left[ \frac{p}{\eta} \right] \frac{u''(W - a - s)}{u'(W - a - s)} \left( \frac{\partial a}{\partial k_A} + \frac{\partial s}{\partial k_A} \right) = \left( r_a \frac{\partial a}{\partial k_A} + r \frac{\partial s}{\partial k_A} \right) \frac{u''(r_a a + rs)}{u'(r_a a + rs)} E_m \left[ \frac{p}{\eta} \right] \quad (15) \\
& + \left( u'(r_a a + rs) \left( r_a \frac{\partial a}{\partial k_A} + r \frac{\partial s}{\partial k_A} \right) - v'(rs) \left( r \frac{\partial s}{\partial k_A} \right) \right) e^{k_R(u(r_a a + rs) - v(rs))} \\
& \times \left( (k_A - k_R) E_m \left[ \frac{p(1-p)}{\eta^2} \right] - k_A E_m \left[ \frac{1-p}{\eta} \right] E_m \left[ \frac{p}{\eta} \right] \right) + \frac{Cov_m \left( \frac{p}{\eta}, \log(\eta) \right)}{k_R}.
\end{aligned}$$

### A.1.2 First-order condition (14)

We now compute the derivative of equation (14) with respect to  $k_A$ . The computation is similar to the one for (13) and we get:

$$\begin{aligned}
& -\frac{u''(W - a - s)}{u'(W - a - s)} E_m \left[ \frac{1-p}{\eta} \right] \left( \frac{\partial a}{\partial k_A} + \frac{\partial s}{\partial k_A} \right) = r \frac{\partial s}{\partial k_A} \frac{v''(rs)}{v'(rs)} E_m \left[ \frac{1-p}{\eta} \right] \quad (16) \\
& - \left( u'(r_a a + rs) \left( r_a \frac{\partial a}{\partial k_A} + r \frac{\partial s}{\partial k_A} \right) - v'(rs) \left( r \frac{\partial s}{\partial k_A} \right) \right) \\
& \times \left( (k_A - k_R) E_m \left[ \frac{p(1-p)}{\eta^2} \right] - k_A E_m \left[ \frac{1-p}{\eta} \right] E_m \left[ \frac{p}{\eta} \right] \right) + \frac{Cov_m \left( \frac{1-p}{\eta}, \log(\eta) \right)}{k_R}.
\end{aligned}$$

### A.1.3 Conclusion

With matrix notation, equations (15) and (16) can be rewritten as:

$$M \begin{pmatrix} r_a \frac{\partial a}{\partial k_A} \\ r \frac{\partial s}{\partial k_A} \end{pmatrix} = \frac{1}{k_R} \begin{pmatrix} Cov_m \left( \frac{p}{\eta}, \log(\eta) \right) \\ Cov_m \left( \frac{1-p}{\eta}, \log(\eta) \right) \end{pmatrix}, \quad (17)$$

with  $M = (M_{ij})_{1 \leq i, j \leq 2}$  and:

$$\begin{aligned} M_{11} &= -\frac{1}{r_a} E_m \left[ \frac{p}{\eta} \right] \frac{u''(W-a-s)}{u'(W-a-s)} - E_m \left[ \frac{p}{\eta} \right] \frac{u''(r_a a + rs)}{u'(r_a a + rs)} + e^{k_R(u(r_a a + rs) - v(rs))} \\ &\quad \times u'(r_a a + rs) \left( k_A E_m \left[ \frac{1-p}{\eta} \right] E_m \left[ \frac{p}{\eta} \right] - (k_A - k_R) E_m \left[ \frac{p(1-p)}{\eta^2} \right] \right), \\ M_{12} &= -\frac{1}{r} E_m \left[ \frac{p}{\eta} \right] \frac{u''(W-a-s)}{u'(W-a-s)} - E_m \left[ \frac{p}{\eta} \right] \frac{u''(r_a a + rs)}{u'(r_a a + rs)} + e^{k_R(u(r_a a + rs) - v(rs))} \\ &\quad \times (u'(r_a a + rs) - v'(rs)) \left( k_A E_m \left[ \frac{1-p}{\eta} \right] E_m \left[ \frac{p}{\eta} \right] - (k_A - k_R) E_m \left[ \frac{p(1-p)}{\eta^2} \right] \right), \\ M_{21} &= -\frac{1}{r_a} E_m \left[ \frac{1-p}{\eta} \right] \frac{u''(W-a-s)}{u'(W-a-s)} \\ &\quad - u'(r_a a + rs) \left( k_A E_m \left[ \frac{1-p}{\eta} \right] E_m \left[ \frac{p}{\eta} \right] - (k_A - k_R) E_m \left[ \frac{p(1-p)}{\eta^2} \right] \right), \\ M_{22} &= -\frac{1}{r} E_m \left[ \frac{1-p}{\eta} \right] \frac{u''(W-a-s)}{u'(W-a-s)} - E_m \left[ \frac{1-p}{\eta} \right] \frac{v''(rs)}{v'(rs)} \\ &\quad - (u'(r_a a + rs) - v'(rs)) \left( k_A E_m \left[ \frac{1-p}{\eta} \right] E_m \left[ \frac{p}{\eta} \right] - (k_A - k_R) E_m \left[ \frac{p(1-p)}{\eta^2} \right] \right). \end{aligned}$$

We also define:

$$\begin{aligned} \kappa &= k_A E_m \left[ \frac{1-p}{\eta} \right] E_m \left[ \frac{p}{\eta} \right] - (k_A - k_R) E_m \left[ \frac{p(1-p)}{\eta^2} \right] \\ &= k_R E_m \left[ \frac{p(1-p)}{\eta^2} \right] - k_A Cov_m \left( \frac{1-p}{\eta}, \frac{p}{\eta} \right). \end{aligned} \quad (18)$$

We compute the determinant of the matrix  $M$ :

$$\begin{aligned}
\det M &= E_m \left[ \frac{1-p}{\eta} \right] E_m \left[ \frac{p}{\eta} \right] \left( \frac{1}{r} - \frac{1}{r_a} \right) \frac{u''(r_a a + rs)}{u'(r_a a + rs)} \frac{u''(W - a - s)}{u'(W - a - s)} \\
&+ E_m \left[ \frac{1-p}{\eta} \right] E_m \left[ \frac{p}{\eta} \right] \left( -\frac{u''(W - a - s)}{r_a u'(W - a - s)} - \frac{u''(r_a a + rs)}{u'(r_a a + rs)} \right) \left( -\frac{v''(rs)}{v'(rs)} \right) \\
&+ \kappa v'(rs) E_m \left[ \frac{p}{\eta} \right] \left( -\frac{1}{r_a} \frac{u''(W - a - s)}{u'(W - a - s)} - \frac{u''(r_a a + rs)}{u'(r_a a + rs)} \right) \\
&+ \kappa u'(r_a a + rs) E_m \left[ \frac{p}{\eta} \right] \left( \frac{1}{r} - \frac{1}{r_a} \right) \left( -\frac{u''(W - a - s)}{u'(W - a - s)} \right) \\
&+ \kappa e^{k_R(u(r_a a + rs) - v(rs))} E_m \left[ \frac{1-p}{\eta} \right] \left( -\frac{1}{r_a} \frac{u''(W - a - s)}{u'(W - a - s)} v'(rs) \right. \\
&\quad \left. + u'(r_a a + rs) \left( \left( \frac{1}{r_a} - \frac{1}{r} \right) \frac{u''(W - a - s)}{u'(W - a - s)} - \frac{v''(rs)}{v'(rs)} \right) \right).
\end{aligned}$$

Since  $r_a \geq r$ , we deduce that  $\det M > 0$  and that the matrix  $M$  is invertible.

We deduce from (17) the expression of partial derivatives  $\frac{\partial s}{\partial k_A}$  and  $\frac{\partial a}{\partial k_A}$ .

$$\begin{pmatrix} r_a \frac{\partial a}{\partial k_A} \\ r \frac{\partial s}{\partial k_A} \end{pmatrix} = \frac{1}{k_R \det M} \begin{pmatrix} M_{22} \text{Cov}_m \left( \frac{p}{\eta}, \log(\eta) \right) - M_{12} \text{Cov}_m \left( \frac{1-p}{\eta}, \log(\eta) \right) \\ -M_{21} \text{Cov}_m \left( \frac{p}{\eta}, \log(\eta) \right) + M_{11} \text{Cov}_m \left( \frac{1-p}{\eta}, \log(\eta) \right) \end{pmatrix}. \quad (19)$$

**Partial derivative  $\frac{\partial s}{\partial k_A}$ .** Equality (19) yields:

$$\begin{aligned}
k_R \det M r \frac{\partial s}{\partial k_A} &= \kappa u'(r_a a + rs) \\
&\times \left( \text{Cov}_m \left( \frac{p}{\eta}, \log(\eta) \right) + e^{k_R(u(r_a a + rs) - v(rs))} \text{Cov}_m \left( \frac{1-p}{\eta}, \log(\eta) \right) \right) \\
&- \frac{u''(r_a a + rs)}{u'(r_a a + rs)} E_m \left[ \frac{p}{\eta} \right] \text{Cov}_m \left( \frac{1-p}{\eta}, \log(\eta) \right) \\
&- \frac{1}{r_a} \frac{u''(W - a - s)}{u'(W - a - s)} \left( -E_m \left[ \frac{1-p}{\eta} \right] \text{Cov}_m \left( \frac{p}{\eta}, \log(\eta) \right) + E_m \left[ \frac{p}{\eta} \right] \text{Cov}_m \left( \frac{1-p}{\eta}, \log(\eta) \right) \right).
\end{aligned} \quad (20)$$

From the definition (10), we deduce:

$$\frac{p}{\eta} = \frac{e^{k_R(u(r_a a + rs) - v(rs))}}{e^{k_R(u(r_a a + rs) - v(rs))} - 1} \frac{1}{\eta} - \frac{1}{e^{k_R(u(r_a a + rs) - v(rs))} - 1}, \quad (21)$$

$$\frac{1-p}{\eta} = \frac{1}{e^{k_R(u(r_a a + rs) - v(rs))} - 1} - \frac{1}{e^{k_R(u(r_a a + rs) - v(rs))} - 1} \frac{1}{\eta}. \quad (22)$$

These two values imply the following equality:

$$\begin{aligned} & Cov_m \left( \frac{p}{\eta}, \log(\eta) \right) + e^{k_R(u(r_a a + rs) - v(rs))} Cov_m \left( \frac{1-p}{\eta}, \log(\eta) \right) \\ &= \frac{e^{k_R(u(r_a a + rs) - v(rs))}}{e^{k_R(u(r_a a + rs) - v(rs))} - 1} \left( Cov_m \left( \frac{1}{\eta}, \log(\eta) \right) - Cov_m \left( \frac{1}{\eta}, \log(\eta) \right) \right) = 0 \end{aligned} \quad (23)$$

and the determinant expression (10) simplifies into:

$$\begin{aligned} k_R \det M r \frac{\partial s}{\partial k_A} &= -\frac{u''(r_a a + rs)}{u'(r_a a + rs)} E_m \left[ \frac{p}{\eta} \right] Cov_m \left( \frac{1-p}{\eta}, \log(\eta) \right) \\ &- \frac{1}{r_a} \frac{u''(W - a - s)}{u'(W - a - s)} \left( -E_m \left[ \frac{1-p}{\eta} \right] Cov_m \left( \frac{p}{\eta}, \log(\eta) \right) + E_m \left[ \frac{p}{\eta} \right] Cov_m \left( \frac{1-p}{\eta}, \log(\eta) \right) \right). \end{aligned} \quad (24)$$

Definitions (21) and (22) imply that  $Cov_m \left( \frac{p}{\eta}, \log(\eta) \right) < 0$  and  $Cov_m \left( \frac{1-p}{\eta}, \log(\eta) \right) > 0$ . Since  $\det M, k_R > 0$ , we obtain from (24) that:  $\frac{\partial s}{\partial k_A} > 0$ .

**Partial derivative  $\frac{\partial a}{\partial k_A}$ .** Using equation (23) stating that  $Cov_m \left( \frac{p}{\eta}, \log(\eta) \right) + e^{k_R(u(r_a a + rs) - v(rs))} Cov_m \left( \frac{1-p}{\eta}, \log(\eta) \right) = 0$ , equality (19) yields:

$$\begin{aligned} k_R \det M r_a \frac{\partial a}{\partial k_A} &= \frac{u''(r_a a + rs)}{u'(r_a a + rs)} E_m \left[ \frac{p}{\eta} \right] Cov_m \left( \frac{1-p}{\eta}, \log(\eta) \right) \\ &- \frac{v''(rs)}{v'(rs)} E_m \left[ \frac{1-p}{\eta} \right] Cov_m \left( \frac{p}{\eta}, \log(\eta) \right) \\ &- \frac{1}{r} \frac{u''(W - a - s)}{u'(W - a - s)} \left( E_m \left[ \frac{1-p}{\eta} \right] Cov_m \left( \frac{p}{\eta}, \log(\eta) \right) - E_m \left[ \frac{p}{\eta} \right] Cov_m \left( \frac{1-p}{\eta}, \log(\eta) \right) \right). \end{aligned} \quad (25)$$

As for (24),  $Cov_m \left( \frac{p}{\eta}, \log(\eta) \right) < 0$  and  $Cov_m \left( \frac{1-p}{\eta}, \log(\eta) \right) > 0$ , which implies with  $k_R \det M r_a > 0$  that:  $\frac{\partial a}{\partial k_A} < 0$ .

**Sum of partial derivatives  $r \frac{\partial s}{\partial k_A} + r_a \frac{\partial a}{\partial k_A}$ .** We deduce from (20) and (25):

$$\begin{aligned} k_R \det M \left( r \frac{\partial s}{\partial k_A} + r_a \frac{\partial a}{\partial k_A} \right) &= -\frac{v''(rs)}{v'(rs)} E_m \left[ \frac{1-p}{\eta} \right] Cov_m \left( \frac{p}{\eta}, \log(\eta) \right) \\ &- \left( \frac{1}{r} - \frac{1}{r_a} \right) \frac{u''(W - a - s)}{u'(W - a - s)} \left( E_m \left[ \frac{1-p}{\eta} \right] Cov_m \left( \frac{p}{\eta}, \log(\eta) \right) - E_m \left[ \frac{p}{\eta} \right] Cov_m \left( \frac{1-p}{\eta}, \log(\eta) \right) \right), \end{aligned}$$

which is negative since  $Cov_m \left( \frac{p}{\eta}, \log(\eta) \right) < 0$  and  $Cov_m \left( \frac{1-p}{\eta}, \log(\eta) \right) > 0$ . We deduce:  $r \frac{\partial s}{\partial k_A} + r_a \frac{\partial a}{\partial k_A} < 0$ .



## A.2 Impact of $k_R$

We will compute the derivatives of (13) and (14) with respect to  $k_R$ , while keeping the ratio  $k_A/k_R$  unchanged. For the sake of clarity, we will denote such a derivative of a function  $f$  as  $\frac{\partial f}{\partial k_R}$  instead of  $\frac{\partial f}{\partial k_R}\Big|_{k_A/k_R}$ . Computing the derivative of (13) yields:

$$\begin{aligned} & -E_m \left[ \frac{p}{\eta} \right] \frac{u''(W-a-s)}{u'(W-a-s)} \left( \frac{\partial a}{\partial k_R} + \frac{\partial s}{\partial k_R} \right) = \left( r_a \frac{\partial a}{\partial k_R} + r \frac{\partial s}{\partial k_R} \right) \frac{u''(r_a a + rs)}{u'(r_a a + rs)} E_m \left[ \frac{p}{\eta} \right] \quad (26) \\ & - \kappa \left( u'(r_a a + rs) \left( r_a \frac{\partial a}{\partial k_R} + r \frac{\partial s}{\partial k_R} \right) - v'(rs) r \frac{\partial s}{\partial k_R} \right) e^{k_R(u(r_a a + rs) - v(rs))} \\ & - \frac{\kappa}{k_R} (u(r_a a + rs) - v(rs)) e^{k_R(u(r_a a + rs) - v(rs))}. \end{aligned}$$

Similarly, computing the derivative of (14) yields:

$$\begin{aligned} & -E_m \left[ \frac{1-p}{\eta} \right] \frac{u''(W-a-s)}{u'(W-a-s)} \left( \frac{\partial a}{\partial k_R} + \frac{\partial s}{\partial k_R} \right) = \beta r^2 \frac{\partial s}{\partial k_R} \frac{v''(rs)}{v'(rs)} E_m \left[ \frac{1-p}{\eta} \right] \quad (27) \\ & + \kappa \left( u'(r_a a + rs) \left( r_a \frac{\partial a}{\partial k_R} + r \frac{\partial s}{\partial k_R} \right) - v'(rs) \left( r \frac{\partial s}{\partial k_R} \right) \right) \\ & + \frac{\kappa}{k_R} (u(r_a a + rs) - v(rs)). \end{aligned}$$

In matrix notation, we deduce from (26) and (27):

$$\begin{pmatrix} r_a \frac{\partial a}{\partial k_R} \\ r \frac{\partial s}{\partial k_R} \end{pmatrix} = \underbrace{\frac{u(r_a a + rs) - v(rs)}{k_R \det M}}_{=\Lambda^{-1} > 0} \kappa \begin{pmatrix} M_{22} & -M_{12} \\ -M_{21} & M_{11} \end{pmatrix} \begin{pmatrix} -e^{k_R(u(r_a a + rs) - v(rs))} \\ 1 \end{pmatrix}. \quad (28)$$

The matrix  $M$  is defined in (17) and  $\det M > 0$ ,  $\kappa > 0$  and  $u(r_a a + rs) - v(rs) > 0$ .

### A.2.1 Partial derivative $\frac{\partial s}{\partial k_R}$ .

Using the expressions of  $M_{11}$  and  $M_{21}$ , we obtain from (28):

$$\begin{aligned} \Lambda r \frac{\partial s}{\partial k_R} &= \frac{1}{r_a} E_m \left[ \frac{1-p}{\eta} \right] \left( -\frac{u''(W-a-s)}{u'(W-a-s)} \right) e^{k_R(u(r_a a + rs) - v(rs))} \quad (29) \\ &+ E_m \left[ \frac{p}{\eta} \right] \left( -\frac{1}{r_a} \frac{u''(W-a-s)}{u'(W-a-s)} - \frac{u''(r_a a + rs)}{u'(r_a a + rs)} \right), \end{aligned}$$

which unambiguously yields:  $\frac{\partial s}{\partial k_R} > 0$ .

### A.2.2 Partial derivative $\frac{\partial a}{\partial k_R}$ .

The expressions of  $M_{12}$  and  $M_{22}$  imply using (28)

$$\begin{aligned} \Lambda r_a \frac{\partial a}{\partial k_R} = & -E_m \left[ \frac{1-p}{\eta} \right] \left( -\frac{1}{r} \frac{u''(W-a-s)}{u'(W-a-s)} - \frac{v''(rs)}{v'(rs)} \right) e^{k_R(u(r_a a + rs) - v(rs))} \\ & - E_m \left[ \frac{p}{\eta} \right] \left( -\frac{1}{r} \frac{u''(W-a-s)}{u'(W-a-s)} - \frac{u''(r_a a + rs)}{u'(r_a a + rs)} \right). \end{aligned} \quad (30)$$

which is negative and implies that:  $\frac{\partial a}{\partial k_R} < 0$ .

### A.2.3 Sum of partial derivatives $r_a \frac{\partial a}{\partial k_R} + r \frac{\partial s}{\partial k_R}$ .

Summing equations (29) and (30) leads to:

$$\begin{aligned} \Lambda \left( r_a \frac{\partial a}{\partial k_R} + r \frac{\partial s}{\partial k_R} \right) = & -E_m \left[ \frac{1-p}{\eta} \right] \left( -\frac{v''(rs)}{v'(rs)} \right) e^{k_R(u(r_a a + rs) - v(rs))} \\ & - \left( \frac{1}{r} - \frac{1}{r_a} \right) E_m \left[ \frac{1-p}{\eta} \right] \left( -\frac{u''(W-a-s)}{u'(W-a-s)} \right) e^{k_R(u(r_a a + rs) - v(rs))} \\ & - \left( \frac{1}{r} - \frac{1}{r_a} \right) E_m \left[ \frac{p}{\eta} \right] \left( -\frac{u''(W-a-s)}{u'(W-a-s)} \right), \end{aligned}$$

which implies since  $r_a \geq r$  that:  $r_a \frac{\partial a}{\partial k_R} + r \frac{\partial s}{\partial k_R} < 0$ .

## B Proof: Corner case: $a = 0$

The annuity choice is zero and the riskless saving choice is determined by the following first-order condition (which is (9) with  $a = 0$ ):

$$\begin{aligned} u'(W-s) = & \beta r u'(rs) \frac{\sum_{i=1}^n m_i p_i \left( p_i + (1-p_i) e^{k_R(u(rs)-v(rs))} \right)^{\frac{k_A}{k_R}-1}}{\sum_{i=1}^n m_i \left( p_i + (1-p_i) e^{k_R(u(rs)-v(rs))} \right)^{\frac{k_A}{k_R}}} \\ & + \beta r v'(rs) e^{k_R(u(rs)-v(rs))} \frac{\sum_{i=1}^n m_i (1-p_i) \left( p_i + (1-p_i) e^{k_R(u(rs)-v(rs))} \right)^{\frac{k_A}{k_R}-1}}{\sum_{i=1}^n m_i \left( p_i + (1-p_i) e^{k_R(u(rs)-v(rs))} \right)^{\frac{k_A}{k_R}}}. \end{aligned} \quad (31)$$

## B.1 Role of $k_A$

Computing the derivative of (31) with respect to  $k_A$  (with  $k_R$  constant) yields after some manipulation:

$$\begin{aligned} -u''(W-s)\frac{\partial s}{\partial k_A} &= \beta r^2 \frac{\partial s}{\partial k_A} \left( u''(rs)E_m \left[ \frac{p}{\eta} \right] + v''(rs)e^{k_R(u(rs)-v(rs))} E_m \left[ \frac{1-p}{\eta} \right] \right) \\ &\quad - \kappa \beta r \left( u'(rs) - v'(rs) \right)^2 e^{k_R(u(rs)-v(rs))} \left( r \frac{\partial s}{\partial k_A} \right) \\ &\quad + \frac{\beta r}{k_R} \left( u'(rs) - v'(rs) \right) Cov_m \left( \frac{p}{\eta}, \log(\eta) \right). \end{aligned}$$

Since  $Cov_m \left( \frac{p}{\eta}, \log(\eta) \right) < 0$ , this proves that  $\frac{\partial s}{\partial k_A}$  has the same sign as  $u'(rs) - v'(rs)$ .

## B.2 Role of $k_R$

Computing the derivative of (31) with respect to  $k_R$  (with  $k_A/k_R$  constant) yields:

$$\begin{aligned} -u''(W-s)\frac{\partial s}{\partial k_R} &= \beta r^2 \frac{\partial s}{\partial k_R} \left( u''(rs)E_m \left[ \frac{p}{\eta} \right] + e^{k_R(u(rs)-v(rs))} v''(rs)E_m \left[ \frac{1-p}{\eta} \right] \right) \\ &\quad - \kappa \beta r \left( u'(rs) - v'(rs) \right)^2 \left( r \frac{\partial s}{\partial k_R} \right) e^{k_R(u(rs)-v(rs))} \\ &\quad - \kappa \frac{\beta r}{k_R} \left( u'(rs) - v'(rs) \right) (u(rs) - v(rs)) e^{k_R(u(rs)-v(rs))}, \end{aligned}$$

which implies that  $\frac{\partial s}{\partial k_R}$  has the same sign as  $u'(rs) - v'(rs)$ .

## C Proof: Corner case $s = 0$

The annuity choice is determined by the following first-order condition – which is (8) with  $a = 0$ :

$$u'(W-a) = \beta r_a u'(r_a a) \frac{\sum_{i=1}^n m_i p_i \left( p_i + (1-p_i) e^{k_R(u(r_a a) - v(0))} \right)^{\frac{k_A}{k_R} - 1}}{\sum_{i=1}^n m_i \left( p_i + (1-p_i) e^{k_R(u(r_a a) - v(0))} \right)^{\frac{k_A}{k_R}}}. \quad (32)$$

## C.1 Role of $k_A$

Computing the derivative of (32) with respect to  $k_A$  (with  $k_R$  constant) yields:

$$\begin{aligned}
 -u''(W - a) \left( \frac{\partial a}{\partial k_A} \right) &= \beta r_a \left( r_a \frac{\partial a}{\partial k_A} \right) u''(r_a a) E_m \left[ \frac{p}{\eta} \right] \\
 &- \kappa \beta r_a u'(r_a a)^2 \left( r_a \frac{\partial a}{\partial k_A} \right) e^{k_R(u(r_a a) - v(0))} + \frac{\beta r_a}{k_R} u'(r_a a) Cov_m \left( \frac{p}{\eta}, \log(\eta) \right),
 \end{aligned} \tag{33}$$

which implies since  $\kappa > 0$  and  $Cov_m \left( \frac{p}{\eta}, \log(\eta) \right) < 0$  that  $\frac{\partial a}{\partial k_A} < 0$ .

## C.2 Role of $k_R$

Computing the derivative of (32) with respect to  $k_R$  (with  $k_R/k_A$  constant) yields:

$$\begin{aligned}
 -u''(W - a - s) \left( \frac{\partial a}{\partial k_R} \right) &= \beta r_a \left( r_a \frac{\partial a}{\partial k_R} \right) u''(r_a a) E_m \left[ \frac{p}{\eta} \right] \\
 &- \kappa \beta r_a u'(r_a a)^2 \left( r_a \frac{\partial a}{\partial k_R} \right) e^{k_R(u(r_a a) - v(0))} \\
 &- \kappa \frac{\beta r_a}{k_R} u'(r_a a)^2 e^{k_R(u(r_a a) - v(0))},
 \end{aligned} \tag{34}$$

which implies since  $\kappa > 0$ :  $\frac{\partial a}{\partial k_R} < 0$ .